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VARIATIONAL PRINCIPLE FOR SHAPED CHARGE JET FORMATION

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A new variational principle for the classical unsolved problem of impinging unequal streams in potential flow is derived for planar flow. A coupled pair of boundary-value problems defined in two flow regions is recovered as the necessary conditions resulting from making stationary a functional associated with the potential energy of the whole flow field. In each region Laplace's equation emerges as the Euler equation, and the derived natural boundary conditions correspond to the appropriate physical boundary conditions on the free surfaces and at the interface between the regions. The application of this result to asymmetric shaped charge jet formation is then discussed. The conservation equations and the optimality conditions for minimum potential energy appear to provide the equations necessary to close the problem.

INTRODUCTION

For most of this century the problem of impinging unequal streams in steady flow has been of interest to workers in fluid dynamics [1–4]. Over the last few decades this problem has become highly relevant in considering asymmetries in the shaped charge jet formation process. The presence of asymmetry brings about the lateral drift of the shaped charge jet particles, causing them to collide on the crater side wall rather than to contribute to the penetration at the bottom of the crater.

Traditionally the problem has been investigated for the case where the densities and speeds (relative to the stagnation point) of the streams are equal, but the widths of the streams differ. There is a solution for non-parallel streams in planar flow in this case. However, in the study of asymmetric jet formation the assumption of equal speeds relative to the stagnation point where the jet is formed is incorrect. Heider and Rottenkolber [5] sought to overcome this problem by choice of frame and achieved a closed form solution, but the agreement with experimental data was not good. The author and co-workers [6–9] have investigated three models with improving, but still not entirely satisfactory, results. In all of these formulations the fundamental difficulty is that there are insufficient equations to solve for all the unknowns. In each case plausible but non-rigorous methods of closing the problem have been attempted by making assumptions about the nature of the flow-field.

For example, the condition that the outgoing jets lie in a straight line was taken as a hypothesis by Pack and Curtis [5] and used in conjunction with the equation of conservation of energy and Bernoulli's law applied to the outgoing jet and slug to close the problem. In fact Curtis and Kelly [6] later showed that the straight-line hypothesis can only be sensibly applied up to a limit that is a function of the incoming stream parameters. The same authors [7] later investigated a formation model based on the concept of a stagnant core around which the fluid moves in circular arcs. The treatment of the flow in the vicinity of the core was approximate. Nonetheless this model exhibited increasingly good agreement with the results of Kinelovskii and Sokolov (loc.cit.) as the angle between the impinging streams grew, coinciding for a head-on collision. Similarly the predictions of the off-axis velocities of shaped charge jets improved with increasing collapse angle. A feature of the model was that the outgoing (generally non-parallel) jets consisted of two adjacent parallel streams, each travelling at the speed of the incoming stream from which it originated. A benefit of this approach is that the free surface conditions are automatically satisfied.

More recently Curtis [9] considered a simplified model in which this feature was retained, but the assumption of the stagnant core was not made. He further assumed that the mean outgoing jet speeds were equal and that the outgoing jets travel in directions close to the nominal axis of symmetry. This set of assumptions resulted in an analytical model that was in better agreement with the experimental data from shaped charges than the earlier models. The model recovered the known special cases correctly and showed that the thinner jet is deflected more than the thicker one – an intuitively pleasing result. However, the agreement with experiment was still not fully satisfactory.

It was desired to improve on this position. Accordingly in this paper a different approach has been adopted in which the underlying physics of the flow is investigated theoretically in the hope that this can be used to find more rigorous models. It was decided to investigate first the associated problem of potential flow, in which the flow is divided into two regions, each region including one of the incoming jets and one of the streams in each of the outgoing jets. A coupled pair of boundary-value problems was formulated and the question of whether a governing variational principle existed was posed. If a minimum energy principle existed, this could explain the existence of the stable solutions observed by Kinelovskii and Sokolov (loc. cit.). The concept of a minimum energy principle had in fact previously been explored by Curtis et al [10] for an approximate flow field with circular streamlines. By contrast, here we consider the true free boundary-value problem, neither constraining the shapes of the free boundaries, nor restricting the treatment to the symmetric case. It is demonstrated that the governing coupled boundary value problem is recovered as the set of necessary conditions for a functional associated with the potential energy to be made stationary. The application of this result to asymmetric shaped charge jet formation is then discussed. It is shown that, in principle, the equations necessary to close the problem emerge naturally as stationary conditions.

BOUNDARY-VALUE PROBLEM

The flow field produced when two streams of incompressible inviscid fluids having unequal speeds, widths and densities meet is investigated. Far from the region of impingement the incoming streams are of speeds U_1, U_2, widths A_1, A_2, and densities ρ_1, ρ_2, respectively. Each stream is divided into two parts, one turning into the jet and one into the slug. The widths of the portions of the first stream turning into the jet and slug are denoted by A_{1J}, A_{1S} respectively. Analogous variables A_{2J}, A_{2S} are defined for the other incoming jet. These widths and the associated jet speeds are preserved in the corresponding outgoing streams because there is no dissipation or compression.

Figure 1 shows the geometrical configuration and the mathematical boundary-value problem in the case where $\rho_1 U_1^2 > \rho_2 U_2^2$. Curtis [9] gives an argument for the interface JCL between the two regions of fluid in the domains D_1 and D_2 respectively containing a cusp C at the point where the stronger incoming first stream is parted by the weaker second. This cusp is a stagnation point for the flow in the region D_2. The streamlines CJ and CL are contact discontinuities on which the normal components of the velocities $\underline{u}_1$ and $\underline{u}_2$ in D_1 and D_2 respectively vanish and the pressure fields p_1 and p_2 are equal. On the outer free surfaces the pressures and normal components of the velocities are zero. Far from the formation region the incoming and outgoing widths and speeds of each component stream are the same.

In the next section a variational principle equivalent to this boundary-value problem is derived.

VARIATIONAL PRINCIPLE

Consider the functional defined over the region $D_1 \cup D_2$ shown in Figure 1 by

$$I[\phi_1,\phi_2;D_1,D_2] = \int_{D_1} \tfrac{1}{2}\rho_1 (U_1^2 - \nabla\phi_1.\nabla\phi_1)\, dA + \int_{D_2} \tfrac{1}{2}\rho_2 (U_2^2 - \nabla\phi_2.\nabla\phi_2)\, dA, \quad (1)$$

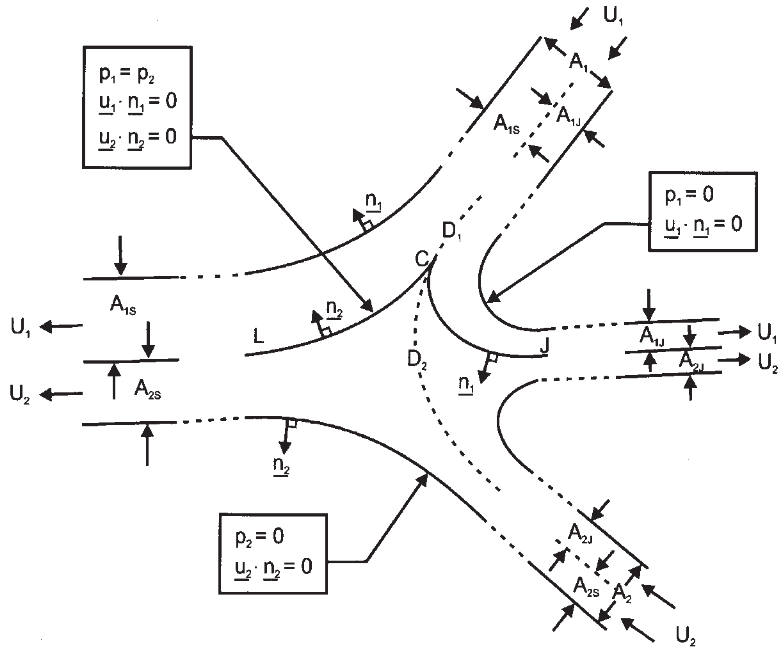

Figure 1. The potential flow field describing the two impinging streams consists of two regions D_1 and D_2. The upper first incoming stream is the stronger of the two, having a higher Bernoulli constant. The cusp point C is a stagnation point for the flow in D_2. The streamlines CJ and CL are contact discontinuities on which the normal components of the velocity fields in each region vanish and the pressures in each region are equal.

where ϕ_1 and ϕ_2 are twice differentiable functions in the spatial co-ordinates x_i, and dA is an element of area. Let ϕ_1 and ϕ_2 satisfy the asymptotic boundary conditions

$$\phi_j \to \pm n_{ji} x_i U_j,\ j = 1,2 \text{ as } r = |\underline{x}| \to \infty . \tag{2}$$

Here the positive and negative signs hold on the outgoing and incoming streams respectively, and the summation convention applies to i only. The outward normals $\underline{n}_j$ from D_1 and D_2 are taken perpendicular to the boundary streamlines.

We seek pairs of functions $\phi_j^{(0)}$, $j = 1, 2$ and domains $D_j^{(0)}$, that make stationary the functional I given by Eq. 1. We consider first-order variations about these functions and domains by writing:

$$\phi_j = \phi_j^{(0)} + \varepsilon\phi_j^{(1)},\quad j = 1, 2, \qquad x_{ji} = x_{ji}^{(0)} + \varepsilon f_j n_{ji},\quad j = 1, 2 \tag{3}$$

where $\varepsilon \ll 1$, the co-ordinates x_{ji} define the boundaries of the domains D_j, $j = 1, 2$, and the functions f_j, $j = 1, 2$ are continuous functions of the co-ordinates $x_{1i}^{(0)}$, $x_{2i}^{(0)}$ defining the boundaries $\partial D_1^{(0)}$, $\partial D_2^{(0)}$ of the domains $D_1^{(0)}$, $D_2^{(0)}$ respectively, as are the normals. The corresponding first-order variation in I is then readily calculated as:

$$\delta I = \sum_{j=1,2} \rho_j \Big(\int_{D_j^{(0)}} \phi_j^{(1)} \nabla^2 \phi_j^{(0)} \, dA_j + \int_{\partial D_j^{(0)}} \Big(\tfrac{1}{2} f_j (U_j^2 - \nabla\phi_j^{(0)} . \nabla\phi_j^{(0)}) \Big) ds_j - \int_{\partial D_j^{(0)}} \Big(\phi_j^{(1)} \underline{n}_j . \nabla\phi_j^{(0)} \Big) ds_j \Big), \tag{4}$$

where ds_j is an element of arc length along the boundary to $D_j^{(0)}$. In deriving Eq. (4) the Divergence Theorem has been applied in both D_1 and D_2. At the stationary solution the right-hand side of Eq. (4) must vanish for all weak variations $\phi_j^{(1)}$, f_j, j = 1,2. Consideration of the first term on the right-hand side of Eq. (4) and standard arguments of the calculus of variations yield the Euler equations:

$$\nabla^2 \phi_j^{(0)} = 0, \qquad j = 1, 2 \tag{5}$$

holding in $D_1^{(0)}$, and $D_2^{(0)}$ respectively. We therefore recover the equations of potential flow in each region as necessary conditions for the stationary solution.

The boundaries $\partial D_1^{(0)}$, $\partial D_2^{(0)}$ comprise the free surfaces, the interface between the regions, and the cross-sections at infinity. Consider the second term on the right-hand side of Eq. (4). On the cross-sections at infinity the integrands vanish as a result of the boundary conditions (2). On the free-surfaces, the arbitrariness of the functions f_j, j = 1, 2 gives the necessary conditions:

$$U_j^2 - \nabla\phi_j^{(0)} . \nabla\phi_j^{(0)} = 0, \qquad j = 1, 2. \tag{6}$$

This is familiar as the pair of conditions that the speeds on the free streamlines remain constant. On the interface it follows from Eq. (3) that $f_2 = -f_1$. The arbitrariness of these functions subject to this constraint yields the necessary condition

$$\tfrac{1}{2} \rho_1 (U_1^2 - \nabla\phi_1^{(0)} . \nabla\phi_1^{(0)}) = \tfrac{1}{2} \rho_2 (U_2^2 - \nabla\phi_2^{(0)} . \nabla\phi_2^{(0)}) \tag{7}$$

In potential flow this condition represents the balance of pressure at the interface between the flows. Finally, consider the third term on the right-hand side of Eq. (4). The arbitrariness of the variations $\phi_j^{(1)}$, j =1,2 on the free surfaces and on the interface between the regions implies the necessary natural boundary conditions that the velocity components normal to the free streamlines and to the interface between the regions D_1 and D_2 vanish. Thus one of

$$\underline{n}_1 . \nabla\phi_1^{(0)} = 0, \qquad \underline{n}_2 . \nabla\phi_2^{(0)} = 0 \tag{8}$$

holds on each free surface as appropriate, while both conditions apply on the interface between the regions. The Dirichlet boundary conditions (2) enforce the vanishing of the contribution to the third integral in Eq. (4) from the cross-sections at infinity as a result of the variations $\phi_j^{(1)}$, j =1,2, being zero there.

We have recovered a pair of coupled boundary value problems describing potential flows with free surfaces in each material. The boundary conditions on the interface between the materials couple the problems. Eqs. (5) are to be solved subject to the free surface conditions (6), boundary conditions (2), and interface conditions (7) and (8). When the functional (1) is evaluated for the solution of the above coupled boundary-value problem, it represents the potential energy associated with the entire flow field. It appears likely that the potential flow solution derived in the preceding section minimises this functional. Note that any functions ϕ_1 and ϕ_2 satisfying the boundary conditions and any suitable domains D_1 and D_2 may be used to furnish approximations to the stationary value of the functional. The choices of the domains $D_1^{(0)}$, $D_2^{(0)}$ as well as the functions $\phi_1^{(0)}$, $\phi_2^{(0)}$, are determined by the variational principle. In other words, the directions of the outgoing streams are determined by the variational principle. It appears likely that this solution is favoured by nature, as evidenced by the work of Kinelovskii and Sokolov (loc. cit.). By skilful choice of the functions ϕ_1 and ϕ_2 and trial domains D_1 and D_2 expressed in terms of a limited number of variable parameters, it may be possible to make accurate estimates of the solution by minimising the functional (1) in terms of those parameters.

APPLICATION TO SHAPED CHARGE JET FORMATION

In the problem of shaped charge jet formation the incoming streams represent the collapsing liner material arriving at the axis of the jet, in a frame moving with the point of formation – the so-called stagnation point. Let the directions of these incoming streams be denoted by θ_1 and θ_2. Let the directions in which the outgoing jet and slug travel be denoted by the unknowns θ_3 and θ_4 respectively. The speeds U_1, U_2, and the widths A_1, A_2, are known, but the divisions A_{1J}, A_{1S} and A_{2J}, A_{2S} are not known. The mass conservation equations are:

$$A_{1J} + A_{1S} = A_1 \,, \qquad A_{2J} + A_{2S} = A_2 \,. \tag{9}$$

These equations allow the elimination of A_{1S}, A_{1S}. We now conjecture that the functional I is expressible as a straightforward function of the unknown parameters A_{1J}, A_{2J}, θ_3, θ_4 and the known parameters describing the flow. Thus

$$I[\phi_1,\phi_2;D_1,D_2] \equiv g(A_{1J},A_{2J},\theta_3,\theta_4;A_1,A_2,U_1,U_2,\theta_1,\theta_2)\,, \tag{10}$$

where the function g is assumed to be smoothly differentiable in the unknown arguments. The hypothesised minimisation associated with the variational principle then yields four equations in the four unknowns, namely:

$$\frac{\partial g}{\partial A_{1J}} = 0, \quad \frac{\partial g}{\partial A_{2J}} = 0, \quad \frac{\partial g}{\partial \theta_3} = 0, \quad \frac{\partial g}{\partial \theta_4} = 0\,. \tag{11}$$

These equations are to be solved simultaneously for the directions θ_3, θ_4, and widths A_{1J}, A_{2J} of the outgoing streams in the jet. Provided that the four equations are indeed independent then the problem is closed and the solution exists. To demonstrate this conclusively it will be necessary to establish the exact form of the function g corresponding to the coupled boundary-value problem derived in the preceding section. The unknown arguments of g in Equation (10) above could provide the variable parameters needed to generate approximate solutions, as discussed above.

CONCLUSIONS

A variational principle equivalent to the coupled boundary-value problem describing the meeting of unequal streams in steady flow has been derived and applied to the problem of shaped charge jet formation. It has been conjectured that the solution corresponds to a minimum of the potential energy. Assuming that the energy functional may be expressed in terms of the parameters describing the flow field at the shaped charge jet formation zone, it has been demonstrated that a set of four equations in the four variables describing the outgoing streams results. If a solution of this set exists then the problem is fully determined. This is in significant contrast to previous models, where various assumptions have been required to achieve closure. It remains to specify the exact form of the potential energy function, or, alternatively, to investigate approximate solutions by exploitation of the underlying variational principle.

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